Online Appendix

Evaluating the Financial Instability Hypothesis: A Positive and Normative Analysis of Leveraged Risk-Taking and Extrapolative Expectations

The Appendix has two parts. The first part proves the propositions and corollaries stated in the text as well as derives the planner's problem in Definition 2. The second part describes the numerical method used to solve the PDEs.

1 Proofs of Propositions and Corollaries

Lemma 1 *At any given time t, households are indifferent among any consumption rate c_t. Moreover, they choose reinvestment rate $i_{2,t}$ and asset holding $k_{2,t}$ as follows:*

$$\mathcal{I}_2'(\iota_{2,t}) = \frac{1}{q_t} \ , \tag{1}$$

and

$$q_t k_{2,t} \left[\begin{array}{ll} = 0 & \text{if } \alpha_{2,t} < 0 \\ \in [0, +\infty) & \text{if } \alpha_{2,t} = 0 \end{array} \right. \ , \tag{2}$$

where the estimated risk-adjusted excess return to allocate the asset to the unproductive technology over holding debt, that is, $\alpha_{2,t} \leq 0$, is given by

$$\alpha_{2,t} \equiv \frac{1}{dt} \hat{E}_t \left[dR_{2,t} \right] - r \leq 0 \ . \tag{3}$$

Proof. Households maximize the present discounted value of consumption

$$B_t \equiv \max_{c_t, \iota_{2,t}, k_{2,t} \geq 0} \hat{E}_t \int_t^{+\infty} e^{-r(s-t)} c_s ds \ , \tag{4}$$

subject to the law of motion of wealth,

$$dw_s = dR_{2,s} q_s k_{2,s} + r(w_s - q_s k_{2,s}) ds - c_s ds + \tau_s ds \ . \tag{5}$$

Let's postulate that

$$B_t = w_t - e^{rt}\int_0^t e^{-rs}\tau_s ds \ . \tag{6}$$

Substituting (4) into (6) and rearranging, one gets the following condition:

$$e^{-rt}w_t + \int_0^t e^{-rs}\left(c_s - \tau_s\right) ds = \hat{E}_t \int_0^{+\infty} e^{-rs} c_s ds \ . \tag{7}$$

The RHS in this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. From applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rw_t = \max_{c_t, \iota_{2,t}, k_{2,t} \geq 0} \left\{ c_t - \tau_t + \left[\frac{1}{dt}\hat{E}_t\left[dR_{2,t}\right] - r\right] q_t k_{2,t} + rw_t - c_t + \tau_t \right\} \ , \tag{8}$$

Note that any c_t is optimal. The optimal $\iota_{2,t}$ and $k_{2,t}$ are

$$\mathcal{I}_2'(\iota_{2,t}) = \frac{1}{q_t} \ , \tag{9}$$

and

$$k_{2,t} \left[\begin{array}{ll} = +\infty & \text{if } \frac{1}{dt}\hat{E}_t\left[dR_{2,t}\right] - r > 0 \\ \in [0, +\infty) & \text{if } \frac{1}{dt}\hat{E}_t\left[dR_{2,t}\right] - r = 0 \\ = 0 & \text{if } \frac{1}{dt}\hat{E}_t\left[dR_{2,t}\right] - r < 0 \end{array} \right. \ . \tag{10}$$

The HJB equation thus reduces to

$$0 = \left[\frac{1}{dt}\hat{E}_t\left[dR_{2,t}\right] - r\right] q_t k_{2,t} \ , \tag{11}$$

where $\iota_{2,t}$ and $k_{2,t}$ are given by (9) and (10), respectively—which under restriction $\alpha_{2,t} \leq 0$, verifies the postulate. ■

Lemma 2 *At any given time t, financiers choose reinvestment rate $\iota_{1,t}$ and asset holding $k_{1,t}$ as follows:*

$$\mathcal{I}_1'(\iota_{1,t}) = \frac{1}{q_t} \ , \tag{12}$$

and

$$\frac{q_t k_{1,t}}{n_t} \left[\begin{array}{ll} = 0 & \textit{if } \alpha_{1,t} < 0 \\ \in [0, \lambda] & \textit{if } \alpha_{1,t} = 0 \\ = \lambda & \textit{if } \alpha_{1,t} > 0 \end{array} \right. \ , \tag{13}$$

where the estimated risk-adjusted excess return to allocate the asset to the productive technology over holding debt, namely, $\alpha_{1,t} \in \mathbb{R}$, is given by

$$\alpha_{1,t} \equiv \frac{1}{dt}\hat{E}_t\left[dR_{1,t}\right] - r + \left(\sigma_{q,t} + \sigma_1\right)\sigma_{v,t} \ . \tag{14}$$

The marginal value of net worth, v_t, satisfies

$$0 = \alpha_{1,t}\frac{q_t k_{1,t}}{n_t} + \mu_{v,t} + \hat{\mu}\omega_t\sigma_{v,t} + \frac{\theta}{v_t} - \theta \ . \tag{15}$$

Proof. Financiers maximize the present discount value of dividend payouts

$$V_t \equiv \max_{\iota_{1,s},k_{1,s}\geq 0} \hat{E}_t \int_t^{\infty} \theta e^{-(r+\theta)(s-t)} n_s ds \ , \tag{16}$$

subject to the law of motion of net worth,

$$dn_s = dR_{1,s} q_s k_{1,s} - r(q_s k_{1,s} - n_s) ds \ , \tag{17}$$

and collateral constraint $q_s k_{1,s} \leq \lambda n_s$, with $n_s \geq 0$.

Note that value $V_t = v_t n_t$ satisfies

$$e^{-(r+\theta)t} v_t n_t + \int_0^t \theta e^{-(r+\theta)s} n_s ds = \hat{E}_t \int_0^{\infty} \theta e^{-(r+\theta)s} n_s ds \ . \tag{18}$$

The RHS of this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. Applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following HJB equation:

$$\begin{aligned} (r+\theta)\, v_t \quad &= \quad \max_{\iota_{1,t},\phi_t\geq 0} \left\{\theta + \left[\mu_{v,t} + \hat{\mu}\omega_t\sigma_{v,t} + \frac{1}{dt}\hat{E}_t\left[dR_{1,t}\right]\phi_t - r\left(\phi_t - 1\right) + \sigma_{v,t}\left(\sigma_{q,t} + \sigma\right)\phi_t\right] v_t\right\} \\ &\quad\quad \text{subject to} : \phi_t \leq \lambda \ . \end{aligned} \tag{19}$$

where $\phi_t \equiv q_t k_{1,t}/n_t$. The optimal $\iota_{1,t}$ and ϕ_t are

$$\mathcal{I}_1'\left(\iota_{1,t}\right) = \frac{1}{q_t} \ . \tag{20}$$

and

$$\phi_t \begin{bmatrix} = \lambda & \text{if } \alpha_{1,t} > 0 \\ \in [0, \lambda] & \text{if } \alpha_{1,t} = 0 \\ = 0 & \text{if } \alpha_{1,t} < 0 \end{bmatrix} . \tag{21}$$

Substituting (20) and (21) into (19), one gets the following equation:

$$\alpha_{1,t}\phi_t + \mu_{v,t} + \hat{\mu}\omega_t\sigma_{v,t} + \frac{\theta}{v_t} - \theta = 0 \,. \tag{22}$$

■

Proposition 1 *Let $\eta_t \equiv n_t/q_t k_t \in [0, 1]$ be the aggregate net worth of financiers as a share of total wealth and let $\kappa_t \equiv k_{1,t}/k_t \in [0, 1]$ be the aggregate share of the asset allocated to the productive technology. Then, the equilibrium outcome is partitioned into the following three regimes,*

$$\begin{array}{llll} \textit{1. Financially unconstrained regime:} & \kappa_t = 1 \leq \lambda\eta_t \,, & \alpha_{1,t} = 0 \,, & \alpha_{2,t} < 0 \,; \\ \textit{2. Financially constrained regime:} & \kappa_t = \lambda\eta_t \in [0,1] \,, & \alpha_{1,t} > 0 \,, & \alpha_{2,t} = 0 \,; \\ \textit{3. Precautionary regime:} & \kappa_t = 0 \,, & \alpha_{1,t} < 0 \,, & \alpha_{2,t} = 0 \,; \end{array} \tag{23}$$

The equilibrium allocation can be summarized as $\{\iota_{1,t}, \iota_{2,t}, \kappa_t\}$, and can be characterized by $\{(1), (3), (12), (14), (22), (23)\}$. The equilibrium utility of households per unit of the asset, namely, $u_t > 0$, satisfies

$$\begin{aligned} 0 \;=\; & \kappa_t \left\{ A_1 - \iota_{1,t} + \left[\mathcal{I}_1(\iota_{1,t}) + \sigma_1 \hat{\mu}\omega_t \right] u_t \right\} + \\ & + (1 - \kappa_t) \left\{ A_2 - \iota_{2,t} + \left[\mathcal{I}_2(\iota_{2,t}) + \sigma_2 \hat{\mu}\omega_t \right] u_t \right\} + \hat{E}_t \left[du_t \right] - r u_t. \end{aligned} \tag{24}$$

Proof. Expressions $\{(1), (3)\}$ characterize the optimality conditions of households and expressions $\{(12), (14), (22)\}$ characterize the optimality conditions of financiers. Expression (23) ensures that market clearing for the asset is consistent with individual optimality. Specifically, if $\alpha_{2,t} < 0$, then $\alpha_{1,t} = 0$ must hold, which requires $\kappa_t = 1$. If $\alpha_{2,t} = 0$, then either $\alpha_{1,t} < 0$ or $\alpha_{1,t} > 0$ must hold. In the first case, $\kappa_t = 0$ is required, while in the second, $\kappa_t = \lambda\eta_t$ is required. Variables $\{\iota_{1,t}, \iota_{2,t}, \kappa_t\}$ together with $c_t/k_t = (A_1 - \iota_{1,t})\,\kappa_t + (A_2 - \iota_{2,t})\,(1 - \kappa_t)$ ensure that market clearing for the good holds. Market clearing for debt automatically holds because of Walras Law.

Let U_t be the utility of households under the equilibrium allocation. Then,

$$U_t \equiv \hat{E}_t \int_t^{+\infty} e^{-r(s-t)} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \ . \tag{25}$$

Utility $\hat{U}_t$ can be expressed as

$$\begin{aligned} & e^{-rt}U_t + \int_0^t e^{-rs} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds = \\ & = \hat{E}_t \int_0^{+\infty} e^{-rs} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1-\kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \ . \end{aligned} \tag{26}$$

The RHS in this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. Applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following HJB equation:

$$0 = \kappa_t \left(A_1 - \iota_{1,t}\right) + \left(1-\kappa_t\right)\left(A_2 - \iota_{2,t}\right) + \hat{E}_t \left[dU_t\right] - rU_t \ . \tag{27}$$

We postulate that $U_t = u_t k_t$, where $u_t > 0$ is an Ito process with disturbance dZ_t. The above HJB equation can then be reduced to

$$\begin{aligned} & 0 = \kappa_t \left\{A_1 - \iota_{1,t} + \left[\mathcal{I}_1(\iota_{1,t}) + \sigma_1 \hat{\mu}\omega_t\right] u_t\right\} + \\ & + \left(1-\kappa_t\right)\left\{A_2 - \iota_{2,t} + \left[\mathcal{I}_2(\iota_{2,t}) + \sigma_2 \hat{\mu}\omega_t\right] \ u_t\right\} + \hat{E}_t\left[du_t\right] - ru_t \ . \end{aligned} \tag{28}$$

■

Proposition 2 *The Markov equilibrium can be analytically characterized as the solution to a system of second-order PDEs for* $\{q, v\}$ *in* $\{\eta, \omega\}$*.*

Proof. This section derives the system of partial differential equations (PDEs) that analytically characterizes the Markov equilibrium. To do so, we consider more general specifications for diagnostic expectations, the collateral constraint, and the production technologies than the baseline specification in Section 2.3 of the paper. Specifically, (i) drift μ_ω can be any function of the state space that does not depend on drift μ_η; (ii) diffusion σ_ω can be any exogenous function of the state space; (iii) expectation weight $\hat{\mu}$ can also be any exogenous function of the state space; (iv) leverage limit λ can be either a parameter or a linear function of value v; and (v) return function $\mathcal{I}_2(\iota_2) \geq 0$ or volatily $\sigma_2 \geq 0$ can be positive. This more general specification suffices to characterize the Markov equilibrium in all of the specifications in the paper. In the remainder of the section, we omit time subscript t.

The equation that determines price q is

$$\begin{array}{ll} \alpha_1 = 0 & \text{if } \kappa = 1 \\ \alpha_2 = 0 & \text{otherwise} \end{array} , \tag{29}$$

or equivalently,

$$\begin{array}{ll} \frac{A_1-\iota_1}{q} + \mu_q + \mathcal{I}_1(\iota_1) + (\sigma_q + \sigma_1)\hat{\mu}\omega + \sigma_q\sigma_1 - r + (\sigma_q + \sigma_1)\sigma_v = 0 & \text{if } \omega \geq \bar{\omega} \text{ and } \eta \geq \bar{\eta} \\ \frac{A_2-\iota_2}{q} + \mu_q + \mathcal{I}_2(\iota_2) + (\sigma_q + \sigma_2)\hat{\mu}\omega + \sigma_q\sigma_2 - r = 0 & \text{otherwise} \end{array} . \tag{30}$$

The equation that determines value v is

$$\alpha_1\phi + \mu_v + \hat{\mu}\omega\sigma_v + \frac{\theta}{v} - \theta = 0 \, . \tag{31}$$

Ito's Lemma implies that for $x \in \{q, v\}$

$$\mu_x = \frac{1}{x}\left[\frac{\partial x}{\partial \omega}\mu_\omega\omega + \frac{\partial x}{\partial \eta}\mu_\eta\eta + \frac{1}{2}\frac{\partial^2 x}{(\partial\omega)^2}(\sigma_\omega\omega)^2 + \frac{\partial^2 x}{\partial\omega\partial\eta}\sigma_\omega\omega\sigma_\eta\eta + \frac{1}{2}\frac{\partial^2 x}{(\partial\eta)^2}(\sigma_\eta\eta)^2\right] , \tag{32}$$

$$\sigma_x = \frac{1}{x}\left[\frac{\partial x}{\partial \omega}\sigma_\omega\omega + \frac{\partial x}{\partial \eta}\sigma_\eta\eta\right] , \tag{33}$$

where recall that

$$\begin{aligned} \mu_\eta &= \left[\frac{A_1 - \iota_1}{q} + \mathcal{I}_1(\iota_1) + \sigma_q\sigma_1\right]\phi + (\mu_q - r)(\phi - 1) - \mu_k \qquad (34) \\ &\quad -\sigma_q\sigma_k + (\sigma_q + \sigma_k)[(\sigma_q + \sigma_k) - \phi(\sigma_q + \sigma_1)] - \left(\theta - \frac{\gamma}{\eta}\right) , \\ \sigma_\eta &= \phi(\sigma_q + \sigma_1) - (\sigma_q + \sigma_k) , \qquad (35) \end{aligned}$$

with

$$\mu_k = \kappa\mathcal{I}_1(\iota_1) + (1-\kappa)\mathcal{I}_2(\iota_2) , \tag{36}$$

$$\sigma_k = \kappa\sigma_1 + (1-\kappa)\sigma_2 \, . \tag{37}$$

According to (32) and (33), objects $\{\mu_q, \sigma_q\}$ depend on $\{\mu_\eta, \sigma_\eta\}$, but according to (34) and (35), objects $\{\mu_\eta, \sigma_\eta\}$ in turn depend on $\{\mu_q, \sigma_q\}$. To eliminate this circularity, we substitute (34) and (35) into (32) and (33). We obtain

$$\mu_\eta = \frac{1}{1-(\phi-1)\,\varepsilon_{q,\eta}} \left\{ \left[\frac{A_1-\iota_1}{q} + (1-\eta)\left[\mathcal{I}_1\left(\iota_1\right)+\sigma_q\sigma_1\right] \right] \phi + \right. \tag{38}$$
$$+ \left(\mu_\omega \varepsilon_{q,\omega} + \xi_{q,\eta/\omega} - r\right)(\phi-1) - (1-\phi\eta)\left[\mathcal{I}_2(\iota_2) - \sigma_k\sigma_2\right] +$$
$$\left. - \left[\sigma_q + \sigma_1\phi\eta + (1-\phi\eta)\sigma_2\right]\left[(\phi-1)\,\sigma_q + (1-\eta)\,\sigma_1\phi\right] - \left(\theta - \frac{\gamma}{\eta}\right) \right\} ,$$

$$\sigma_\eta = \frac{(\phi-1)\,\sigma_\omega\varepsilon_{q,\omega} + (1-\eta)\,\sigma_1\phi - (1-\phi\eta)\sigma_2}{1-(\phi-1)\,\varepsilon_{q,\eta}} , \tag{39}$$

where

$$\varepsilon_{q,\eta} \equiv \frac{\partial q}{\partial \eta}\frac{\eta}{q} , \quad \varepsilon_{q,\omega} \equiv \frac{\partial q}{\partial \omega}\frac{\omega}{q} , \tag{40}$$

$$\xi_{q,\eta/\omega} \equiv \frac{1}{2}\frac{\partial^2 q}{(\partial\omega)^2}(\sigma_\omega\omega)^2 + \frac{\partial^2 q}{\partial\eta\partial\omega}\sigma_\eta\eta\sigma_\omega\omega + \frac{1}{2}\frac{\partial^2 q}{(\partial\eta)^2}\left(\sigma_\eta\eta\right)^2 . \tag{41}$$

Thus, the system of equations,

$$\left\{(30)\,,\ (31)\,,\ (32)\,,\ (33)\,,\ (37)\,,\ (38)\,,\ (39)\right\} , \tag{42}$$

with

$$\iota_j = \mathcal{I}_j'^{-1}\left(\frac{1}{q}\right) , \tag{43}$$

$$\kappa = \phi\eta \text{ with } \phi = \min\left\{\lambda, \frac{1}{\eta}\right\}\mathbf{1}_{\omega\geq\bar{\omega}} , \tag{44}$$

$$\bar{\omega}\left(\eta\right) = \left\{\omega < 0 : \alpha_1\left(\omega,\eta\right) = \alpha_2\left(\omega,\eta\right) = 0\right\} , \tag{45}$$

$$\bar{\eta}\left(\omega\right) = \left\{\eta \in [0,1] : \lambda\left(\omega,\eta\right)\eta = 1\right\} , \tag{46}$$

determines a second-order PDEs for $\{q, v\}$ in $\{\omega, \eta\}$.

We impose the following boundary conditions to the PDEs:

$$\lim_{\eta\to 1}\sigma_q = 0\ ,\quad \lim_{\eta\to 1}\frac{\partial\sigma_q}{\partial\eta} = 0\ ,\quad \lim_{\eta\to 1}\sigma_v = 0\ ,\quad \lim_{\eta\to 1}\frac{\partial\sigma_v}{\partial\eta} = 0\ . \tag{47}$$

These conditions ensure that diffusions σ_q and σ_v vanish smoothly as the aggregate net worth of financiers approaches total wealth. ■

Corollary 1 *In the economy with rational expectations and without financial frictions, neither sentiment ω nor wealth share η influence the equilibrium outcome. The asset price is a constant that satisfies*

$$\alpha_1 = 0 \Leftrightarrow \frac{A_1 - \iota_1}{q} + \mathcal{I}_1(\iota_1) - r = 0\ ,\ \textit{with}\ \ \mathcal{I}_1'(\iota_1) = \frac{1}{q}\ . \tag{48}$$

Value $v = 1$ is also a constant. The aggregate quantity of the asset is allocated to the productive technology, that is, $\kappa = 1$. The social value of the asset equals the asset price, that is, $u = q$.

Proof. Under RE, the productive technology yields higher return than the unproductive one, accordingly the Equilibrium relationship #3 from (23) cannot occur. In the absence of financial frictions, the allocation of the asset to the productive technology is not restricted by the collateral constraint, hence the Equilibrium relationship #2 cannot occur. Accordingly, the conditions Equilibrium relationship #1 characterize the equilibrium, that is $\alpha_1 = 0$ with $\kappa = 1$. It derives that $v = 1$, since financiers earn no rent on the asset, and

$$\alpha_1 = \frac{1}{dt} E\left[dR_1\right] - r = 0.$$

Finally, the price of the asset q is constant and satisfies

$$\frac{A_1 - \iota_1}{q} + I_1(\iota_1) - r = 0,$$

where ι_1 satisfies (20). ■

Corollary 2 *In the economy with diagnostic expectations and without financial frictions, sentiment ω is the only relevant state that affects the equilibrium outcome. A threshold state $\bar{\omega} < 0$ exists such that*

$$\begin{array}{ll} \textit{if}\ \omega < \bar{\omega} \Rightarrow & \kappa = 0\ ,\ \ \alpha_1 < 0\ ,\ \ \alpha_2 = 0\ ; \\ \textit{if}\ \omega > \bar{\omega} \Rightarrow & \kappa = 1\ ,\ \ \alpha_1 = 0\ ,\ \ \alpha_2 < 0\ ; \end{array}\ . \tag{49}$$

The threshold state $\bar{\omega} < 0$ is the solution to

$$\alpha_1 = \alpha_2 = 0 \Rightarrow \frac{A_1 - \iota_1 - A_2}{q} + \mathcal{I}_1(\iota_1) + (\sigma_q + \hat{\mu}\omega)\,\sigma_1 = 0\ . \tag{50}$$

Proof. In the absence of financial frictions, Equilibrium relationship #2 cannot occur. Accordingly, the economy alternates between Equilibrium relationships #1 and #3, depending on the value of sentiment ω, i.e., depending on the perceived relative returns to each technology, as indicated in (49). The sentiment threshold state $\bar{\omega}$ is such that the perceived return to each technology is the same, i.e., $\alpha_1 = \alpha_2$. Using (3) and (14), one gets (50). ■

Corollary 3 *In the economy with rational expectations and financial frictions, wealth share η is the only relevant state that affects the equilibrium outcome. A threshold state $\bar{\eta} \in (0,1)$ exists such that*

$$\begin{array}{lll} \textit{if } \eta < \bar{\eta} \Rightarrow & \kappa = \lambda\eta < 1 \ , & \alpha_1 > 0 \ , \ \ \alpha_2 = 0 \ ; \\ \textit{if } \eta > \bar{\eta} \Rightarrow & \kappa = 1 \ , & \alpha_1 = 0 \ , \ \ \alpha_2 < 0 \ ; \end{array} . \tag{51}$$

The threshold state $\bar{\eta} \in (0,1)$ is $\bar{\eta} = \frac{1}{\lambda}$.

Proof. Under rational expectations, only Equilibrium relationships #1 and #2 can occur, since the productive technology is correctly perceived as providing higher returns. Accordingly, the economy alternates between the financially constrainted and financially unconstrained regime, depending on the wealth share η of financiers. The cut-off value $\bar{\eta}$ naturally satisfies $\bar{\eta} = \frac{1}{\lambda}$. ■

Corollary 4 *In the economy with diagnostic expectations and financial frictions, both sentiment ω and wealth share η affect the equilibrium outcome. Thresholds $\bar{\omega} < 0$ and $\bar{\eta} \in (0,1)$, partition the state space as follows:*

$$\begin{array}{lll} \textit{if } \omega < \bar{\omega} \Rightarrow & \kappa = 0 \ , & \alpha_1 < 0 \ , \ \ \alpha_2 = 0 \ ; \\ \textit{if } \omega > \bar{\omega} \textit{ and } \eta < \bar{\eta} \Rightarrow & \kappa = \lambda\eta \ , & \alpha_1 > 0 \ , \ \ \alpha_2 = 0 \ ; \\ \textit{if } \omega > \bar{\omega} \textit{ and } \eta > \bar{\eta} \Rightarrow & \kappa = 1 \ , & \alpha_1 = 0 \ , \ \ \alpha_2 < 0 \ ; \end{array} \tag{52}$$

Threshold process $\bar{\omega}$ is the solution to

$$\alpha_1 = \alpha_2 = 0 \Rightarrow \frac{A_1 - \iota_1 - A_2}{q} + \mathcal{I}_1(\iota_1) + (\sigma_q + \hat{\mu}\omega)\,\sigma_1 + (\sigma_q + \sigma_1)\,\sigma_v = 0 \ , \tag{53}$$

and the threshold state $\bar{\eta} \in (0,1)$ is $\bar{\eta} = \frac{1}{\lambda}$.

Proof. With both diagnostic expectations and financial frictions, the economy alternates between the three Equilibrium relationships. The characterization of cut-off states $\bar{\omega}$ and $\bar{\eta}$ follows from the proofs of Propositions 6 and 7. ■

Proposition 3 *If agents rely on Ito path $\{dX_s\}_{s<t}$ to form diagnostic expectations about Ito variable dY_t, the implied diagnostic expectation operator over disturbance dZ_t is*

$$\hat{E}_t\left[dZ_t\right] = \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt \ , \tag{54}$$

where $\sigma_{Y,t} \in \mathbb{R}$ is the diffusion of the variable and where sentiment $\omega_t \in \mathbb{R}$ is given by

$$\omega_t = \int_0^t e^{-\delta(t-s)}dX_s \ . \tag{55}$$

Corollary 5 *If $dX_s = \frac{dR_{1,s}-\hat{E}_s[dR_{1,s}]}{Std_s[dR_{1,s}]}$ and $dY_t = dR_{1,t}$, then*

$$\hat{E}_t\left[dZ_t\right] = \hat{\mu}\omega_t dt \ , \tag{56}$$

and

$$d\omega_t = \left(-\delta + \frac{\hat{\mu}}{\sigma_{q,t}+\sigma_1}\right)\omega_t dt + dZ_t \ . \tag{57}$$

Proof. Let ω_t be a sentiment operator tied to a Ito process $\{dX_s\}$, i.e.,

$$\omega_t = \int_0^t e^{-\delta(t-s)}dX_s. \tag{58}$$

A diagnostic operator over a generic Ito process dY_t is defined as:

$$\hat{E}_t\left[dY_t\right] \equiv E_t\left[d\hat{Y}_t\right], \quad \text{with} \quad d\hat{Y}_t \equiv \hat{\mu}\omega_t dt + dY_t \tag{59}$$

Let's define the expectation operator $\check{E}_t[dZ_t]$ as

$$\check{E}_t[dZ_t] \equiv E_t[d\hat{Z}_t], \text{ with } d\hat{Z}_t \equiv \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt + dZ_t \ . \tag{60}$$

Then

$$\check{E}_t[dY_t] = \check{E}_t[\mu_{Y,t}Y_t dt + \sigma_{Y,t}Y_t dZ_t] = \mu_{Y,t}Y_t dt + \sigma_{Y,t}Y_t\check{E}_t[dZ_t] = \left(\mu_{Y,t}Y_t + \hat{\mu}\omega_t\right)dt \ . \tag{61}$$

Accordingly, $\check{E}_t[dY_t] = \hat{E}_t[dY_t]$, and the implied diagnostic expectation operator over dZ_t is

$$\hat{E}_t\left[dZ_t\right] = \check{E}_t[dZ_t] = \hat{\mu}\frac{\omega_t}{\sigma_{Y,t}Y_t}dt \ . \tag{62}$$

Applying these results to the specific case $dX_s = \frac{dR_{1,s} - \hat{E}_s[dR_{1,s}]}{Std_s[dR_{1,s}]}$ and $dY_t = dR_{1,t}$, one gets the expressions presented in Corollary 2. ■

Proposition 4 *The socially optimal reinvestment rate solves*

$$\mathcal{I}_1'(\iota_1) = \frac{1 + \frac{1}{1-(\phi-1)\varepsilon_{q,\eta}} \frac{1}{q} \frac{\partial \tilde{u}}{\partial \eta}}{\tilde{u} + \frac{1-\eta}{1-(\phi-1)\varepsilon_{q,\eta}} \frac{\partial \tilde{u}}{\partial \eta}} \ . \tag{63}$$

The socially optimal share κ maximizes the RHS in (1). The candidate solutions are $\kappa = 0$, $\kappa = \min\{\lambda\eta, 1\}$, and any interior $\kappa \in (0, \min\{\lambda\eta, 1\})$ that solves

$$\begin{aligned} 0 \quad = \quad & \left[\frac{A_1 - \iota_1 - A_2}{\tilde{u}} + \mathcal{I}_1(\iota_1) + (\sigma_{\tilde{u}} + \tilde{\mu}\omega)\,\sigma_1\right] + \varepsilon_{\tilde{u},\eta}\left[\frac{\partial \mu_\eta}{\partial \kappa} + (\tilde{\mu}\omega + \kappa\sigma_1)\,\frac{\partial \sigma_\eta}{\partial \kappa}\right] + \\ & + \frac{1}{\tilde{u}}\left(\frac{\partial^2 \tilde{u}}{(\partial \eta)^2}\sigma_\eta \eta + \frac{\partial^2 \tilde{u}}{\partial \eta \partial \omega}\right)\frac{\partial \sigma_\eta}{\partial \kappa}\eta \ , \end{aligned} \tag{64}$$

where $\frac{\partial \mu_\eta}{\partial \kappa}$ and $\frac{\partial \sigma_\eta}{\partial \kappa}$ are the partial derivatives of μ_η and σ_η with respect to κ, respectively.

Proof. This proof lay outs and solves the problem of the planner. The present discounted value of consumption under expectation weight $\tilde{\mu} \in [0, \hat{\mu}]$ is

$$\tilde{U}_t \equiv \tilde{E}_t \int_t^{+\infty} e^{-r(s-t)} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1 - \kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \ , \tag{65}$$

where expectation operator $\tilde{E}_t[\cdot]$ is

$$\tilde{E}_t[dZ_t] \equiv E_t[d\tilde{Z}_t], \text{ with } d\tilde{Z}_t \equiv \tilde{\mu}\omega_t dt + dZ_t \ . \tag{66}$$

Note that the term in brackets in the integrand follows from resource constraint

$$c_t = y_t = \kappa_t \left(A_1 - \iota_{1,t}\right) + \left(1 - \kappa_t\right)\left(A_2 - \iota_{2,t}\right) . \tag{67}$$

Utility $\tilde{U}_t$ can be expressed as

$$\begin{aligned} & e^{-rt}\tilde{U}_t + \int_0^t e^{-rs} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1 - \kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \\ & = \tilde{E}_t \int_0^{+\infty} e^{-rs} \left[\kappa_s \left(A_1 - \iota_{1,s}\right) + \left(1 - \kappa_s\right)\left(A_2 - \iota_{2,s}\right)\right] k_s ds \ . \end{aligned} \tag{68}$$

The RHS in this equation is the conditional expectation of a random variable. Thus, the drift of the RHS is null. Applying Ito's Lemma to the LHS and equalizing the resulting drift process to zero, one gets the following HJB equation:

$$0 = \kappa_t (A_1 - \iota_{1,t}) + (1-\kappa_t)(A_2 - \iota_{2,t}) + \tilde{E}_t \left[d\tilde{U}_t \right] - r\tilde{U}_t \; . \tag{69}$$

We postulate that $U_t = u_t k_t$, where $u_t > 0$ is an Ito process with disturbance dZ_t. The above HJB equation can then be reduced to

$$\begin{aligned} 0 &= \kappa_t \{A_1 - \iota_{1,t} + [\mathcal{I}_1(\iota_{1,t}) + \sigma_1 \tilde{\mu}\omega_t] \tilde{u}_t\} + \\ &+ (1-\kappa_t)\{A_2 - \iota_{2,t} + [\mathcal{I}_2(\iota_{2,t}) + \sigma_2 \tilde{\mu}\omega_t] \tilde{u}_t\} + \tilde{E}_t [d\tilde{u}_t] - r\tilde{u}_t \; . \end{aligned} \tag{70}$$

In what follows, we restrict attention to a Markov structure with same state variables as in the competitive equilibrium. Thus, we omit time subscript $_t$ from now on. In addition, we consider the parametrization of the baseline specification (Section 2.3 of the paper). Equation (70) can then be expressed as

$$\begin{aligned} r\tilde{u} &= \kappa\{A_1 - \iota_1 + [\mathcal{I}_1(\iota_1) + \sigma_1\tilde{\mu}\omega]\tilde{u}\} + (1-\kappa) A_2 + \frac{\partial \tilde{u}}{\partial \omega}(-\delta\omega + \tilde{\mu}\omega + \kappa\sigma_1) + \\ &+ \frac{\partial \tilde{u}}{\partial \eta}(\mu_\eta \eta + \sigma_\eta \eta \tilde{\mu}\omega + \sigma_\eta \eta \kappa \sigma_1) + \frac{1}{2}\frac{\partial^2 \tilde{u}}{(\partial\omega)^2} + \frac{\partial^2 \tilde{u}}{\partial\omega\partial\eta}\sigma_\eta \eta + \frac{1}{2}\frac{\partial^2 \tilde{u}}{(\partial \eta)^2}(\sigma_\eta \eta)^2 \Big\} \; , \end{aligned} \tag{71}$$

where

$$\begin{aligned} \mu_\eta &= \frac{1}{1-(\frac{\kappa}{\eta}-1)\varepsilon_{q,\eta}} \left\{ \left[\frac{A_1 - \iota_1}{q} + \mathcal{I}_1(\iota_1) + \sigma_q \sigma_1 \right] \frac{\kappa}{\eta} - \kappa \mathcal{I}_1(\iota_1) - \sigma_q \kappa \sigma_1 + \right. \\ &+ \frac{1}{q}\left[-\frac{\partial q}{\partial \omega}\delta\omega + \frac{1}{2}\frac{\partial^2 q}{(\partial \omega)^2} + \frac{\partial^2 q}{\partial\omega\partial\eta}\sigma_\eta \eta + \frac{1}{2}\frac{\partial^2 q}{(\partial\eta)^2}(\sigma_\eta\eta)^2 - rq \right]\left(\frac{\kappa}{\eta} - 1\right) + \\ &+ (\sigma_q + \kappa\sigma_1)\left[(\sigma_q + \kappa\sigma_1) - \frac{\kappa}{\eta}(\sigma_q + \sigma_1) \right] - \left. \left(\theta - \frac{\gamma}{\eta}\right) \right\} \; , \end{aligned} \tag{72}$$

$$\sigma_\eta = \frac{\frac{\kappa}{\eta}\left(\frac{1}{q}\frac{\partial q}{\partial \omega} + \sigma_1\right) - \left(\frac{1}{q}\frac{\partial q}{\partial\omega} + \kappa\sigma_1\right)}{1 - \left(\frac{\kappa}{\eta} - 1\right)\varepsilon_{q,\eta}} \; , \tag{73}$$

with

$$\sigma_q = \frac{\left(\frac{\kappa}{\eta} - \kappa\right)\varepsilon_{q,\eta}\sigma_1 + \frac{1}{q}\frac{\partial q}{\partial \omega}}{1 - \left(\frac{\kappa}{\eta} - 1\right)\varepsilon_{q,\eta}} \; . \tag{74}$$

The above formulae for $\{\mu_\eta, \sigma_\eta, \sigma_q\}$ follow from evaluating $\{(33)\,,\ (38)\,,\ (39)\}$ at the baseline parametrization.

The problem of the planner is then

$$r\tilde{u} = \max_{\{\iota_1,\kappa\}} \left\{ \kappa \left\{ A_1 - \iota_1 + \left[\mathcal{I}_1(\iota_1) + \sigma_1 \tilde{\mu}\omega \right] \tilde{u} \right\} + (1-\kappa) A_2 + \frac{\partial \tilde{u}}{\partial \omega} \left(-\delta\omega + \tilde{\mu}\omega + \kappa\sigma_1 \right) + \right. \tag{75}$$
$$\left. + \frac{\partial \tilde{u}}{\partial \eta} \left(\mu_\eta \eta + \sigma_\eta \eta \tilde{\mu}\omega + \sigma_\eta \eta \kappa \sigma_1 \right) + \frac{1}{2} \frac{\partial^2 \tilde{u}}{(\partial\omega)^2} + \frac{\partial^2 \tilde{u}}{\partial\omega\partial\eta} \sigma_\eta \eta + \frac{1}{2} \frac{\partial^2 \tilde{u}}{(\partial\eta)^2} \left(\sigma_\eta \eta \right)^2 \right\} ,$$

with

$$\iota_1 \in [0, A_1] \text{ and } \kappa \in [0, \min\{\lambda\eta, 1\}] , \tag{76}$$

where $\{\mu_\eta, \sigma_\eta, \sigma_q\}$ are given by $\{(72), (73), (74)\}$.

The first-order condition with respect to ι_1 implies that

$$\mathcal{I}_1'(\iota_1) = \frac{1 + \frac{1}{1-(\phi-1)\varepsilon_{q,\eta}} \frac{1}{q} \frac{\partial \tilde{u}}{\partial \eta}}{\tilde{u} + \frac{1-\eta}{1-(\phi-1)\varepsilon_{q,\eta}} \frac{\partial \tilde{u}}{\partial \eta}} . \tag{77}$$

Note that the problem is concave in ι_1.

The first-order condition with respect to κ implies that

$$\left[\frac{A_1 - \iota_1 - A_2}{\tilde{u}} + \mathcal{I}_1(\iota_1) + (\sigma_{\tilde{u}} + \tilde{\mu}\omega)\sigma_1 \right] + \varepsilon_{\tilde{u},\eta} \left[\frac{\partial \mu_\eta}{\partial \kappa} + (\tilde{\mu}\omega + \kappa\sigma_1) \frac{\partial \sigma_\eta}{\partial \kappa} \right] +$$
$$+ \frac{1}{\tilde{u}} \left(\frac{\partial^2 \tilde{u}}{\partial\omega\partial\eta} + \frac{\partial^2 \tilde{u}}{(\partial\eta)^2} \sigma_\eta \eta \right) \frac{\partial \sigma_\eta}{\partial \kappa} \eta \gtreqless 0 , \tag{78}$$

with $\kappa = 0$ if inequality "$<$" holds and $\kappa = \min\{\lambda\eta, 1\}$ if the other inequality does so.

Note that $\frac{\partial \mu_\eta}{\partial \kappa}$ and $\frac{\partial \sigma_\eta}{\partial \kappa}$ are the partial derivatives with respect to κ of the RHS on expressions (72) and (73), respectively. Diffusion $\sigma_{\tilde{u}}$ is

$$\sigma_{\tilde{u}} = \frac{1}{\tilde{u}} \left[\frac{\partial \tilde{u}}{\partial \omega} + \frac{\partial \tilde{u}}{\partial \eta} \sigma_\eta \eta \right] . \tag{79}$$

■

Proposition 5 *The socially optimal allocation and its associated mappings $\{\tilde{u}, v, q\}$ is analytically characterized by a system of second-order PDEs for the mappings in the state $\{\omega, \eta\}$.*

Proof. The equations that determine price q and value v are

$$\begin{cases} \frac{A_1-\iota_1}{q}+\mu_q+\mathcal{I}_1\left(\iota_1\right)+\left(\sigma_q+\sigma_1\right)\hat{\mu}\omega+\sigma_q\sigma_1-r+\left(\sigma_q+\sigma_1\right)\sigma_v=0 & \text{if } \kappa=1 \\ \frac{A_2}{q}+\mu_q+\sigma_q\hat{\mu}\omega-r=0 & \text{otherwise} \end{cases} . \tag{80}$$

and

$$\alpha_1\frac{\kappa}{\eta}+\mu_v+\hat{\mu}\omega\sigma_v+\frac{\theta}{v}-\theta=0\ , \tag{81}$$

respectively.

The system of equations,

$$\left\{(32)\,,\ (33)\,,\ (84)\,,\ (72)\,,\ (73)\,,\ (77)\,,\ (78)\,,\ (80)\,,\ (81)\right\}\ , \tag{82}$$

thus determines a second-order PDEs for $\{\tilde{u},q,v\}$ in $\{\omega,\eta\}$.

We impose the following boundary conditions to the PDEs:

$$\lim_{\eta\to 1}\sigma_x=0\ ,\ \ \lim_{\eta\to 1}\frac{\partial\sigma_x}{\partial\eta}=0\ . \tag{83}$$

for $x\in\{\tilde{u},q,v\}$. ■

Corollary 6 *In the economy with diagnostic expectations and without financial frictions, the socially optimal allocation is first-best efficient according to the expectation weight of the planner. If the planner is benevolent, the socially optimal allocation is the same as the equilibrium allocation. If the planner is paternalistic, the socially optimal allocation is the same as the equilibrium allocation of the economy presented in subsection 3.1.*

Proof. If the planner is benevolent, the socially optimal allocation is the same as the equilibrium allocation.

Let's postulate that $\partial\tilde{u}/\partial\eta=0$. Thus,

$$\begin{aligned} r\tilde{u} &= \max_{\{\iota_1,\kappa\}}\left\{\kappa\left\{A_1-\iota_1+\left[\mathcal{I}_1(\iota_1)+\sigma_1\hat{\mu}\omega\right]\tilde{u}\right\}+\left(1-\kappa\right)A_2+\right. \\ &\qquad\left.+\frac{\partial\tilde{u}}{\partial\omega}\left(-\delta\omega+\tilde{\mu}\omega+\kappa\sigma_1\right)+\frac{1}{2}\frac{\partial^2\tilde{u}}{\left(\partial\omega\right)^2}\right\}\ . \end{aligned} \tag{84}$$

Let's also postulate that $\tilde{u}=q$. Then, the equilibrium allocation solves the optimization problem, which verifies the postulates.

If the planner is paternalistic, the socially optimal allocation is the same as the equilibrium allocation of the economy presented in subsection 3.1.

Let's postulate that $\partial\tilde{u}/\partial\eta = \partial\tilde{u}/\partial\omega = 0$. Thus,

$$r\tilde{u} \quad = \quad \max_{\{\iota_1,\kappa\}} \left\{\kappa\left\{A_1 - \iota_1 + \mathcal{I}_1(\iota_1)\tilde{u}\right\} + (1-\kappa)\, A_2\right\}.$$

Let's also postulate that $\tilde{u}$ equals the asset price of the economy presented in subsection 3.1. Then, the equilibrium allocation of that economy solves the optimization problem, which verifies the postulates. ■

2 Numerical Solution Method

To solve the PDEs we use spectral methods. Specifically, we interpolate $\{q, v\}$ or $\{\tilde{u}, q, v\}$ with linear combinations of Chebyshev polynomials of the first kind. We evaluate the interpolation at the Chebyshev nodes. We use a nonlinear solver to find the coefficients associated with the polynomials in the linear combination. As initial guess for the solver, we use the values of $\{q, v\}$ or $\{\tilde{u}, q, v\}$ in the economy of Section 3.1 of the paper.